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Dynamics of the SK model in the spin-glass phase

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Abstract. We have first solved numerically the mean field equations of the SK model obtained with Parisi's ansatz for a wide range of values of the temperature and the magnetic field. Then, using Sompolinsky's formulation of the spin-glass dynamics that shows that the finite time spin correlations decay algebrically with a power $\nu(T,h)$, we obtain the exact behaviour $\nu(T,h=0)$ down to $T/T_c = 0.07$ and the much weaker dependence on the magnetic field for several values of the temperature.

1. Introduction

It is now an accepted fact that for spin glasses the spin–spin correlation function has an algebraic decay for long times: $C(t) \sim t^{-\nu(T,h)}$, but we do not know very much about the dynamical exponent ν yet. For 3D spin glasses, Monte Carlo simulations [1] suggest that for low temperatures ν tends to some small value compatible with zero, while for the SK model the simulations [2] suggest that the value of ν must be close to $\frac{1}{2}$ in all the spin-glass phase (in the absence of an external magnetic field); the only attempt to calculate this exponent for the SK model [3, 4] has been done using the replica symmetric static solution.

In this paper we present the exact computation of $\nu(T,h)$ for the SK model in the glassy phase. We find that the introduction of the replica symmetry breaking (RSB) solution reduces visibly the value of the dynamic exponent in the range $T \leq 0.7\ T_c$, but it does not change dramatically the qualitative behaviour; in particular, for low temperatures ν tends to a finite value (0.37...), quite different from zero.

2. The static SK model

The SK model [5] is a magnetic system composed by N Ising spins $\{S_i = \pm 1, i = 1, \ldots, N\}$, with a Hamiltonian

$$\mathcal{H}_{[J]}\{S\} = -\frac{1}{2}\sum_{i\neq j} J_{ij} S_i S_j - h\sum_i S_i \tag{2.1}$$

where the parameters $\{J_{ij}\}$ are chosen at random from the distribution

$$\mathrm{P}(J_{ij}) = \sqrt{\frac{N}{2\pi J^2}}\exp\left(-\frac{N J_{ij}^2}{2J^2}\right). \tag{2.2}$$

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The replica solution gives the free energy density of the SK model, defined as the average over the probability distribution of the J's of the quantity

$$f_{[J]} \equiv -\frac{1}{N\beta} \ln \sum_{\{S\}} \exp\left(-\beta \mathcal{H}_{[J]}\{S\}\right). \tag{2.3}$$

It can be proven [5, 2] that

$$f \equiv \int \cdots \int \left[\prod_{i<j} dJ_{ij} \mathrm{P}(J_{ij})\right] f_{[J]} \tag{2.4}$$

is given by

$$f = -\frac{1}{\beta} \lim_{n \to 0^+} \left\{ \frac{\beta^2 J^2}{4} \left(1 - \frac{2}{n} \sum_{a<b} Q_{ab}^2\right) + \frac{1}{n} \ln \mathrm{Tr}_n \exp \mathcal{L}[Q] \right\} \tag{2.5}$$

with

$$\mathcal{L}[Q] = \beta^2 J^2 \sum_{a<b} Q_{ab} S^a S^b. \tag{2.6}$$

Tr_n denotes the sum over all the possible configurations of n Ising spins $\{S^a, a = 1, \ldots, n\}$.

For integer n, (Q_{ab}) is an $n \times n$ matrix the elements of which must minimize the free energy density in the $n \to 0^+$ limit.

Parisi's ansatz for the Replica Solution [6, 7, 8] gives the parametrization for integer n that supplies the correct solution of the SK model. The expression of the replica matrix Q for integer n is obtained as the $K \to \infty$ limit of block matrices $Q^{(K)}$, defined in terms of the two families of parameters, $\{q_i \in \mathbb{R}, i = 1, \ldots, K\}$ and $\{m_j \in \mathbb{N}, j = 1, \ldots, K + 1\}$, by

$$Q_{ab} = q_i \qquad \text{if} \begin{cases} \mathrm{Int}\left(\dfrac{a}{m_i}\right) \neq \mathrm{Int}\left(\dfrac{b}{m_i}\right) \\ \mathrm{Int}\left(\dfrac{a}{m_{i+1}}\right) = \mathrm{Int}\left(\dfrac{b}{m_{i+1}}\right). \end{cases} \tag{2.7}$$

In the $n \to 0^+$ limit, the two sequences of parameters $\{q_i\}$ and $\{m_j\}$—that supply the dimension of the jth block in the matrix $Q^{(K)}$—can be unified defining a function

$$q(x) = q_i \qquad \text{when } m_i \leq x < m_{i+1} \tag{2.8}$$

that, in the $K \to \infty$, $n \to 0^+$ limit, becomes a continuous function in the interval $x \in [0, 1]$.

With this ansatz, the expression for the free-energy density of the system is [9]:

$$-\beta f(\beta, h) = \frac{\beta^2 J^2}{4} \left(1 - 2q(1) + \int_0^1 q^2(x)\right) + \int_{-\infty}^{+\infty} \frac{\mathrm{d}y}{J\sqrt{2\pi}} \exp\left(\frac{(y-h)^2}{2J^2 q(0)}\right) g(0, y) \tag{2.9}$$

where $q(x)$ and $m(x,y) \equiv J\, \partial g(x,y)/\partial y$ verify a set of coupled integro-differential equations that can be written as

$$q(x) = \int_{-\infty}^{+\infty} \mathrm{d}y\, P(x,y)\, m(x,y)^2 \tag{2.10a}$$

$$G(x,y;\tilde{x},\tilde{y}) = [2\pi(q(\tilde{x}) - q(x))J^2]^{-1/2} \exp\left(-\frac{(y-\tilde{y})^2}{2J^2[q(\tilde{x}) - q(x)]}\right) \tag{2.10b}$$

$$m(x,y) = \int_{-\infty}^{+\infty} \mathrm{d}\tilde{y}\; G(x,y;1,\tilde{y}) \tanh(\beta\tilde{y}) + \int_x^1 \mathrm{d}\tilde{x}\, \beta J^2 \dot{q}(\tilde{x})\, \tilde{x} \times \int_{-\infty}^{+\infty} \mathrm{d}\tilde{y} G(x,y;\tilde{x},\tilde{y}) m(\tilde{x},\tilde{y})\, m'(\tilde{x},\tilde{y}) \tag{2.10c}$$

$$P(x,y) = (2\pi q(x)J^2)^{-1/2} \exp\left(-\frac{(y-h)^2}{2q(x)J^2}\right) - \int_0^x \mathrm{d}\tilde{x}\, \beta J^2 \dot{q}(\tilde{x})\, \tilde{x} \times \int_{-\infty}^{+\infty} \mathrm{d}\tilde{y}\, G(\tilde{x},\tilde{y};x,y)[m(\tilde{x},\tilde{y})P(\tilde{x},\tilde{y})]' \tag{2.10d}$$

where we have used the notation $\dot{a}(x) \equiv \mathrm{d}a(x)/\mathrm{d}x$ and $b'(x,y) \equiv \partial b(x,y)/\partial y$.

In the first part of this work we have found the numerical solution of these equations.

3. The dynamical SK model

We will now consider a set of N continuous spins $\{-\infty < S_i < +\infty, i = 1, \ldots, N\}$, with the Hamiltonian

$$\beta\mathcal{H}'_{[J]}\{S\} = \sum_i \left[V(S_i) - \beta h_i S_i\right] - \frac{\beta}{2} \sum_{i \neq j} J_{ij} S_i S_j \tag{3.1}$$

where $V(S) = r_0 S^2 + uS^4$, and $r_0 < 0$ and $u > 0$ may be chosen constants independent of β; the Ising-spin limit is obtained sending $r_0 \to -\infty$ and $u \to +\infty$ with $|r_0|/u \to 2$.

For a dynamical model we need a set of equations of motion with the properties:

(i) the system must approach the equilibrium;

(ii) we want to simulate the random termic agitation of the spins.

The easiest choice is

$$\tau_0 \frac{\partial S_i(t)}{\partial t} = -\frac{\partial(\beta\mathcal{H}'_{[J]})}{\partial S_i} + \xi_i(t) \tag{3.2}$$

where τ_0 is the characteristic microscopic time of the spins and the $\{\xi_i\}$ are a set of random Gaussian noises with the properties

$$\langle \xi_i(t) \rangle_\xi = 0$$
$$\langle \xi_i(t)\xi_j(t') \rangle_\xi = 2\tau_0 \delta_{ij} \delta(t-t'). \tag{3.3}$$

Now we have a set of N coupled differential equations of motions, yet it is possible to decouple them performing the average over the J's.

Performing that average [3,4] we obtain for the equation of motion the expression

$$\tau_0 \frac{\partial S_i(t)}{\partial t} = -\frac{\partial V(S_i)}{\partial S_i} + \beta z + \beta \eta_i(t) + \beta h + \xi_i(t) + \beta \int_{-\infty}^{t} \mathrm{d}t'\, S_i(t')\, \mathcal{G}(t-t') \qquad (3.4)$$

where

(i) $$\mathcal{G}(t-t') = \frac{1}{N}\sum_i \mathcal{G}_i(t-t') \equiv \frac{1}{N}\sum_i \frac{\delta \langle S_i(t)\rangle}{\delta h_i(t')}$$

is the linear response function;

(ii) z is a random magnetic field that follows the equilibrium internal distribution of magnetic fields $\mathcal{P}(z)$, which is obtained from the function $P(x,y)$ of the static solution: $\mathcal{P}(z) = P(1,z)$;

(iii) the η_i are a set of random Gaussian noises that verify

$$\begin{aligned} \langle \eta_i(t)\rangle_\eta &= 0 \\ \langle \eta_i(t)\eta_i(t')\rangle_\eta &= \mathcal{C}_i(t-t') \end{aligned} \qquad (3.5)$$

in which $\mathcal{C}_i(t-t') \equiv \langle S_i(t)S_i(t')\rangle - \langle S_i(\infty)\rangle^2$ is the spin autocorrelation function;

(iv) $h = \lim_{t\to\infty} h_i(t)$; this limit must be site-independent for the validity of the above statement about the distribution $\mathcal{P}(z)$;

(v) the last term in (3.4) is caused by the indirect effect of $S_i(t')$ on $S_i(t)$: the ith spin polarizes the other spins at the time t' and the reaction of the those spins is sensible also at the time t.

It is worth noting that in this equation the direct effect on the ith spin of the other $N-1$ spins has been substituted by a random local magnetic field $\zeta_i = z + \eta_i$, where z is nothing but the continuum component of this noise and η_i is his fluctuating part.

From this equation, Sompolinsky and Zippelius [3, 4] obtained the following information about the autocorrelation function $\mathcal{C}(t) \equiv 1/N \sum_i \mathcal{C}_i(t)$:

(i) $T > T_c(h) \Longrightarrow \mathcal{C}(t) \sim \exp(-t/\tau(T))$ when $t \gg \tau_0$, and $\tau(T) \sim (T-T_c)^{-1}$ when $T \to T_c^+$;

(ii) $T \le T_c(h) \Longrightarrow \mathcal{C}(t) \sim t^{-\nu(T,h)}$ when $t \gg \tau_0$ where, in the limit of Ising spins, $\nu(T,h)$ verifies the following equation:

$$\frac{\pi \cot(\pi\nu)}{B(\nu,\nu)} = \frac{[m^2(1-m^2)^2]}{[(1-m^2)^3]} \qquad (3.6)$$

where in (3.6) we have used the notation

$$\begin{aligned} B(\nu,\nu) &\equiv \Gamma(\nu)^2/\Gamma(2\nu) \\ [\mathcal{A}] &\equiv \int_{-\infty}^{+\infty} \mathrm{d}z\, \mathcal{P}(z)\mathcal{A}(z) \\ m(z) &\equiv \tanh\big(\beta(z-h)\big) \end{aligned} \qquad (3.7)$$

but they were unable to obtain the correct behaviour of $\nu(T,h)$ in the spin-glass phase because they did not use the RSB solution for the distribution $\mathcal{P}(z)$; solving the equations obtained from Parisi's ansatz, we can now obtain the internal distribution $\mathcal{P}(z)$ and then the dynamical exponent $\nu(T,h)$.

4. Results

In all the numerical calculations we have set $J = 1$, that is equivalent to measuring all the temperatures and magnetic fields in units of $T_c(0) = J$.

4.1. Thermodynamical test for the solutions

From the numerical solutions obtained with Parisi's ansatz, we can immediately calculate the entropy density of the SK model as [10]:

$$\frac{S}{Nk} \equiv -\beta^2 \frac{\partial f(\beta,h)}{\partial \beta} = -\frac{\beta^2}{4}\left(1-q(1)\right)^2 + \int_{-\infty}^{+\infty} \mathrm{d}z\, \mathcal{P}(z)[\ln(2\cosh\beta z) - \beta z \tanh\beta z]. \tag{4.1}$$

This computation is very important because the other ansatz proposed for the replica solution failed in this test, giving a negative entropy density for low temperatures. Instead, our solutions give a low temperature behaviour for the entropy density of the type

$$\frac{S(T,h=0)}{Nk} = \alpha T^2 + \mathcal{O}(T^3) \tag{4.2}$$

and we have obtained $\alpha = 0.71 \pm 0.01$, where the uncertainty is due to the fact that, as of now, we have the solutions only down to $T = 0.07$. In figure 1 we can see the behaviour of the entropy density and the reduced entropy $s(T) \equiv S/(NkT^2)$ in the absence of an external magnetic field.

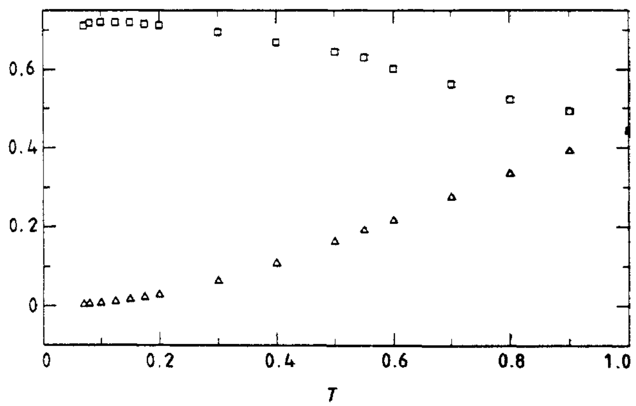

Figure 1. Behaviour of the entropy density (triangles) and of the reduced entropy (squares) for $h = 0$.

4.2. Static results

As a by-product of our calculations we have obtained all the interesting static quantities of the SK model.

In particular, for the field distribution $\mathcal{P}(z) = P(1,z)$ our results are in perfect agreement with the previous numerical solutions [9] in the range ($T \geq 0.20$) studied in that work. As was expected, the value of the distribution at the origin vanishes when the temperature is lowered; the results are compatible with a linear dependence of $\mathcal{P}(0)$ from T:

$$\mathcal{P}(0) = \gamma T + \mathcal{O}(T^2) \qquad \text{with} \qquad \gamma = 0.73 \pm 0.02. \tag{4.3}$$

Another very important quantity that we have calculated is the order parameter function $q(x)$ for all the values of the temperature and magnetic field studied: we have found that it verifies a functional form of the type

$$q(x,T,h) = \begin{cases} q_0(T,h) & \text{for } 0 \le x \le x_0(T,h) \\ Q(x/T) & \text{for } x_0(T,h) < x < x_1(T) \\ q_1(T) & \text{for } x_1(T) \le x \le 1. \end{cases} \tag{4.4}$$

In figure 2 we can see the values obtained for the scale function $Q(x/T)$ for many different values of the temperature and the magnetic field: we can conclude that the scaling law for the order parameter function is valid (to the accuracy of the solutions) in all the spin-glass phase.

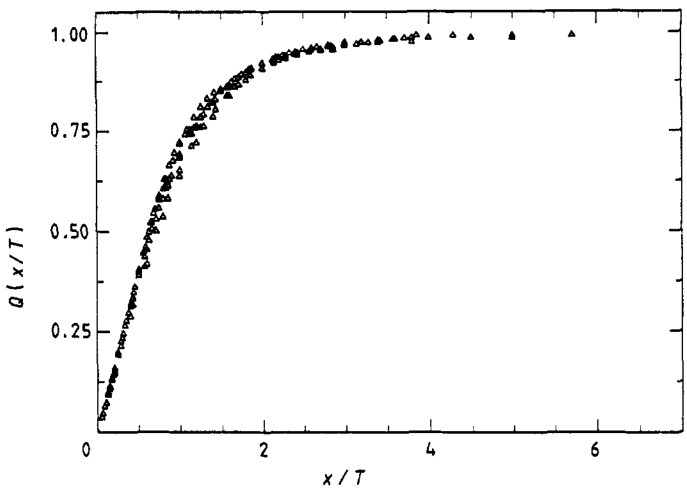

Figure 2. Scale function $Q(x/T)$.

4.3. The dynamical exponent $\nu(T,h)$

Finally, we give the results obtained for the dynamical exponent introduced in section 3. The first thing that is worth noting is that the replica symmetric solution used in [3, 4] gives a result in which ν is independent of h. This result is exact along the De Almeida–Thouless line, where equation (2.10) has trivial solutions. With RSB we can compute ν for all temperatures and magnetic fields in the glass phase; in figure 3 we show two of the results: in the left graph we see the behaviour of $\nu(h=0)$ where the effect of RSB is more visible, and in the right graph is shown the dependence $\nu(h)$

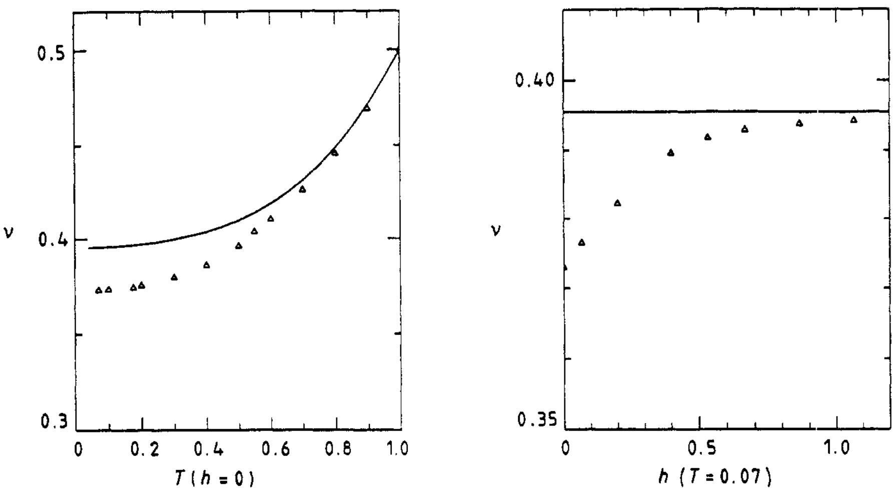

Figure 3. Two of the results for the dynamical exponent $\nu(T,h)$; in both graphs, the continuous line shows the results obtained without RSB, while the triangles are the exact result.

for $T = 0.07$, where we can see how approaching the critical line the effect of RSB vanishes.

From all these results we can deduce the following conclusions:

(i) $\nu(T,h)$ increases with the temperature in all the spin-glass phases:

$$\nu(T_1,h) < \nu(T_2,h) \qquad \forall\, 0 \le T_1 < T_2 \le T_c(h) \tag{4.5}$$

(ii) $\nu(T,h)$ also increases with the magnetic field:

$$\nu(T,h_1) < \nu(T,h_2) \qquad \forall\, 0 \le h_1 < h_2 \le h_c(T) \tag{4.6}$$

(iii) in all the glass phases $\nu(T,h)$ depends only slightly on the magnetic field; in fact, we can see that there are no dramatic differences between the behaviour of the exponent down the De Almeida–Thouless line and in zero magnetic field (this also means that the introduction of RSB does not drastically change the results): for example we have that

$$\lim_{T\to 0} \nu(T,0)\Big|_{\mathrm{RSB}} = 0.3726 \pm 0.0004 \tag{4.7a}$$

while

$$\lim_{T\to 0} \nu(T,0)\Big|_{\mathrm{not\ RSB}} = \lim_{T\to 0} \nu(T,h_c(t))\Big|_{\mathrm{RSB}} = 0.39529 \pm 0.00005 \tag{4.7b}$$

(iv) the behaviour obtained for $\nu(T,h=0)$ shows that the simulations of [2] are not completely valid (although ν becomes almost independent of the temperature in the range $T \le 0.3$), but deeply disagree from the results found [1] in the dynamical Monte Carlo simulations of three-dimensional spin glasses with only nearest-neighbour interactions, suggesting in this way that the upper critical dimension for spin glasses is greater than three.

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